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LETTER TO THE EDITOR

Approximate symmetries and approximate solutions for a multidimensional Landau–Ginzburg equation

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Abstract. We give the approximate symmetries for the multidimensional Landau–Ginzburg equation $\sum_{i=1}^{3}\partial^2 u/\partial x_i^2 + \partial u/\partial x_4 = a_1 + a_2 u + \varepsilon u^n$ where $n \in \mathcal{R}$ and $0 < \varepsilon \ll 1$. We also construct approximate solutions for this nonlinear equation using the approximate symmetries.

The problem of generalizing asymptotic methods (Krylov–Bogolubov method) for solving nonlinear partial differential equations is well documented in the literature (Bogolubov and Mitropol'skii 1974). The asymptotic method is mainly applied to partial differential equations in two independent variables (Kevorkian and Cole 1980). Group-theoretical reduction can be used to reduce the number of independent variables of a multidimensional partial differential equation. After this reduction the asymptotic method can be applied to the reduced equations.

Recently the concept of approximate symmetry was introduced (Fushchich and Shtelen 1989, Baikov *et al* 1989). Within this approach it is possible to obtain approximate solutions of a given multidimensional partial differential equation (Mitropol'skii and Shul'ga 1988). The main steps of this method are the following:

(1) Find the Lie symmetries of the given partial differential equation whereby the number of independent variables (dimensions) may be reduced by group-theoretical methods.

(2) Find an approximate system of partial differential equations for the given partial differential equation or for the dimensionally reduced partial differential equation.

(3) Find the Lie symmetries of the approximate system such that the approximate system can be reduced to a partial differential equation with two independent variables or an ordinary differential equation.

(4) The approximate reduced system can then be investigated to obtain exact solutions. If exact solutions cannot be found for such a (nonlinear) system the asymptotic method can be applied.

Consider the general system of nonlinear partial differential equations

$$F_\nu\left(x_i, u_j, \frac{\partial u_j}{\partial x_i}, \ldots, \frac{\partial^r u_j}{\partial x_{i_1}\partial x_{i_2}\ldots\partial x_{i_r}}\right) = 0. \qquad (1)$$

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We assume that F_ν is a smooth function with respect to the arguments, where $\nu = 1, \ldots, q$, $i = 1, \ldots, m$, $j = 1, \ldots, n$ and $r = i_1 + i_2 + \cdots + i_r$.

We represent the solution of (1) in the following form:

$$u_j = u_{(0)j} + \varepsilon u_{(1)j} + \varepsilon^2 u_{(2)j} + O(\varepsilon^3)\,. \tag{2}$$

Substituting (2) into (1) and separating out with respect to ε (up to ε^2) we obtain the coupled system of partial differential equations

$$F_{(0)\nu}\left(x_i, u_{(0)j}, \frac{\partial u_{(0)j}}{\partial x_i}, \ldots, \frac{\partial^r u_{(0)j}}{\partial x_{i_1}\partial x_{i_2}\ldots\partial x_{i_r}}\right) = 0 \tag{3}$$

$$F_{(1)\nu}\left(x_i, u_{(0)j}, u_{(1)j}, \frac{\partial u_{(0)j}}{\partial x_i}, \frac{\partial u_{(1)j}}{\partial x_i}, \ldots, \frac{\partial^r u_{(0)j}}{\partial x_{i_1}\partial x_{i_2}\ldots\partial x_{i_r}}, \frac{\partial^r u_{(1)j}}{\partial x_{i_1}\partial x_{i_2}\ldots\partial x_{i_r}}\right) = 0 \tag{4}$$

$$F_{(2)\nu}\left(x_i, u_{(0)j}, u_{(1)j}, u_{(2)j}, \frac{\partial u_{(0)j}}{\partial x_i}, \frac{\partial u_{(1)j}}{\partial x_i}, \frac{\partial u_{(2)j}}{\partial x_i}, \ldots, \frac{\partial^r u_{(0)j}}{\partial x_{i_1}\partial x_{i_2}\ldots\partial x_{i_r}}, \frac{\partial^r u_{(1)j}}{\partial x_{i_1}\partial x_{i_2}\ldots\partial x_{i_r}}, \frac{\partial^r u_{(2)j}}{\partial x_{i_1}\partial x_{i_2}\ldots\partial x_{i_r}}\right) = 0\,. \tag{5}$$

Definition 1. A Lie point symmetry of the coupled systems (3) and (4), which approximates (1) is a first-order approximate symmetry of (1).

Definition 2. A Lie point symmetry of the coupled systems (3), (4) and (5), which approximates (1) is a second-order approximate symmetry of (1).

We investigate the first-order approximate symmetry of the multidimensional Landau–Ginzburg equation

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} + \frac{\partial u}{\partial x_4} = a_1 + a_2 u + \varepsilon u^n \tag{6}$$

which describes the kinetics of phase transitions where a_1 and a_2 are real constants and $0 < \varepsilon \ll 1$. Equation (6) admits the following Lie symmetry vector fields (Euler and Steeb 1992) where we consider two different cases.

Case 1. For arbitrary $a_1, a_2, n \in \mathcal{R}$ the Lie algebra is spanned by the seven vector fields (translations and rotations)

$$Z_1 = \frac{\partial}{\partial x_1} \qquad Z_2 = \frac{\partial}{\partial x_2} \qquad Z_3 = \frac{\partial}{\partial x_3} \qquad Z_4 = \frac{\partial}{\partial x_4}$$

$$Z_5 = x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1} \qquad Z_6 = x_2\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_2} \qquad Z_7 = x_3\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_3}\,.$$

Case 2. For arbitrary n with $a_1 = a_2 = 0$ the Lie algebra is spanned by the vector fields $Z_1, \ldots, Z_7$ and the scaling vector field

$$Z_8 = \frac{1}{2}(1-n)\sum_{i=1}^{3} x_i \frac{\partial}{\partial x_i} + (1-n)x_4 \frac{\partial}{\partial x_4} + u\frac{\partial}{\partial u}. \tag{7}$$

We construct the approximate system for (6). Let $u = u_0 + \varepsilon u_1$. Since we only have one dependent variable, we set $u_{(0)1} = u_0$, $u_{(1)1} = u_1$. The approximate system of partial differential equations is then given by

$$\begin{aligned} \frac{\partial^2 u_0}{\partial x_1^2} + \frac{\partial^2 u_0}{\partial x_2^2} + \frac{\partial^2 u_0}{\partial x_3^2} + \frac{\partial u_0}{\partial x_4} &= a_1 + a_2 u_0 \\ \frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2} + \frac{\partial u_1}{\partial x_4} &= a_2 u_1 + u_0^n . \end{aligned} \tag{8}$$

For arbitrary n and arbitrary a_1, a_2 the Lie algebra for system (8) is spanned by the symmetry vector fields $Z_1, \ldots, Z_7$ and

$$Z_8 = f(x)\frac{\partial}{\partial u_1} \tag{9}$$

where $x = (x_1, x_2, x_3, x_4)$ and f satisfies the linear partial differential equation

$$\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2} + \frac{\partial f}{\partial x_4} = a_2 f. \tag{10}$$

For the scaling symmetry of system (8) we consider the following cases:

Case 1. For $a_1 \neq 0$ and $a_2 \neq 0$ system (8) is not scale-invariant.

Case 2. For $a_1 \neq 0$ and $a_2 = 0$ system (8) admits the following scaling vector field:

$$Z_9 = \sum_{i=1}^{3} x_i \frac{\partial}{\partial x_i} + 2x_4 \frac{\partial}{\partial x_4} + 2u_0 \frac{\partial}{\partial u_0} + 2(n+1)u_1 \frac{\partial}{\partial u_1} \tag{11}$$

together with $Z_1, \ldots, Z_8$, where Z_8 is given by (9).

Case 3. For $a_1 = 0$ and $a_2 \neq 0$ system (8) admits the following symmetry vector fields:

$$Z_9 = u_0 \frac{\partial}{\partial u_0} + n u_1 \frac{\partial}{\partial u_1} \tag{12}$$

$$Z_{10} = (f(x) + u_0)\frac{\partial}{\partial u_1} \tag{13}$$

together with $Z_1, \ldots, Z_8$, where Z_8 is given by (9). Here f must satisfy (10). Z_9 is a scaling symmetry.

Case 4. For $a_1 = 0$ and $a_2 = 0$ system (8) admits the following scaling vector field:

$$Z_9 = (1 - nb)\sum_{i=1}^{3} x_i \frac{\partial}{\partial x_i} + 2(1 - nb)x_4\frac{\partial}{\partial x_4} + 2(1 + b)u_0\frac{\partial}{\partial u_0} + 2(1 + n)u_1\frac{\partial}{\partial u_1} \tag{14}$$

together with $Z_1, \ldots, Z_8$, where Z_8 is given by (9). Here b is an arbitrary real constant.

We give the Lie symmetry vector fields for (10) in order to find exact solutions of (10). Our aim is to obtain some exact forms for the symmetry vector field Z_9 of system (8). From the Lie symmetry vector field ansatz

$$Z = \sum_{i=1}^{4} \xi_i(x, f)\frac{\partial}{\partial x_i} + \eta(x, f)\frac{\partial}{\partial f}$$

we find that the smooth coefficient functions $\xi_1, \xi_2, \xi_3, \xi_4, \eta$ are given by

$$\xi_1 = bx_1 + g_1x_4 + c_{12}x_2 + c_{13}x_3 + d_1 \qquad \xi_2 = bx_2 + g_2x_4 - c_{12}x_1 + c_{23}x_3 + d_2$$

$$\xi_3 = bx_3 + g_3x_4 - c_{13}x_1 - c_{23}x_2 + d_3 \qquad \xi_4 = 2bx_4 + d_0$$

$$\eta = \left[\tfrac{1}{2}(g_1x_1 + g_2x_2 + g_3x_3) + 2a_2bx_4 + c\right] f\,.$$

Here $c_{12}, c_{23}, c_{13}, g_i, d_j, b$ ($i = 1, \ldots, 3$; $j = 0, \ldots, 3$) and c are real constants with $c_{12}^2 + c_{23}^2 + c_{13}^2 = 1$. With the help of these Lie symmetry vector fields we find the following exact solutions for (10):

$c = 1$:

$$f(x) = (\alpha \cdot x)\exp(a_2x_4) \tag{15}$$

$c = -1$:

$$f(x) = x_4^{-1/2}\exp\left(a_2x_4 + \frac{(\alpha \cdot x)^2}{4x_4}\right) \tag{16}$$

$c = -2$:

$$f(x) = x_4^{-3/2}(\alpha \cdot x)\exp\left(a_2x_4 + \frac{(\alpha \cdot x)^2}{4x_4}\right) \tag{17}$$

$c = a_2$:

$$f(x) = [c_1(\alpha \cdot x) + c_2]\exp(a_2x_4) \tag{18}$$

$$f(x) = [c_1(x^2)^{-1/2} + c_2]\exp(cx_4) \tag{19}$$

$a_2 > c$:

$$f(x) = \{c_1\exp[\sqrt{a_2 - c}(\alpha \cdot x)] + c_2\exp[-\sqrt{a_2 - c}(\alpha \cdot x)]\}\exp(cx_4) \tag{20}$$

$a_2 < c$:

$$f(x) = \{c_1 \cos[\sqrt{c-a_2}(\alpha \cdot \boldsymbol{x})] + c_2 \sin[\sqrt{c-a_2}(\alpha \cdot \boldsymbol{x})]\} \exp(c x_4) \tag{21}$$

$a_2 \neq c$:

$$f(x) = c_1 \exp(a_2 x_4) . \tag{22}$$

Here c_1 and c_2 are real arbitrary constants, $\boldsymbol{x} := (x_1, x_2, x_3)$ and

$$\alpha \cdot \boldsymbol{x} := \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 \qquad \alpha^2 = 1 .$$

We now obtain approximate solutions for (6) by finding exact solutions for system (8). For the case $a_1 = 0$ with a_2 and n arbitrary we note that $u_0 = 0$ satisfies the first equation of system (8) trivially. By inserting this solution into the second equation of system (8) we obtain (10) with $f = u_1$. Approximate solutions for (6) thus take the form

$$u(x) = \varepsilon f(x) \tag{23}$$

where different functions for f are given by (15)–(22).

In order to investigate other exact solutions for system (8), using the Lie symmetry vector fields listed above, we make the following ansätze for system (8):

$$u_0(x) = \varphi_1(\omega(x)) f_1(x) + g_1(x) \tag{24}$$

$$u_1(x) = \varphi_2(\omega(x)) f_2(x) + g_2(x) \tag{25}$$

where $\omega = (\omega_1, \omega_2, \omega_3)$. It follows that

$$\sum_{j=1}^{3} \frac{\partial \varphi_1}{\partial \omega_j} \frac{\partial \omega_j}{\partial x_4} f_1 + \varphi_1 \frac{\partial f_1}{\partial x_4} + \frac{\partial g_1}{\partial x_4} + \left[\sum_{j=1}^{3} \frac{\partial^2 \varphi_1}{\partial \omega_j^2} \sum_{i=1}^{3} \left(\frac{\partial \omega_j}{\partial x_i} \right)^2 + \sum_{j=1}^{3} \frac{\partial \varphi_1}{\partial \omega_j} \Delta \omega_j \right.$$
$$\left. + 2 \sum_{(j<k)=1}^{3} \frac{\partial^2 \varphi_1}{\partial \omega_j \partial \omega_k} \sum_{i=1}^{3} \frac{\partial \omega_j}{\partial x_i} \frac{\partial \omega_k}{\partial x_i} \right] f_1 + 2 \sum_{i=1}^{3} \frac{\partial f_1}{\partial x_i} \sum_{j=1}^{3} \frac{\partial \varphi_1}{\partial \omega_j} \frac{\partial \omega_j}{\partial x_i}$$
$$+ \varphi_1 \Delta f_1 + \Delta g_1 - a_1 - a_2 [\varphi_1 f_1 + g_1] = 0 \tag{26}$$

and

$$\sum_{j=1}^{3} \frac{\partial \varphi_2}{\partial \omega_j} \frac{\partial \omega_j}{\partial x_4} f_2 + \varphi_2 \frac{\partial f_2}{\partial x_4} + \frac{\partial g_2}{\partial x_4} + \left[\sum_{j=1}^{3} \frac{\partial^2 \varphi_2}{\partial \omega_j^2} \sum_{i=1}^{3} \left(\frac{\partial \omega_j}{\partial x_i} \right)^2 + \sum_{j=1}^{3} \frac{\partial \varphi_2}{\partial \omega_j} \Delta \omega_j \right.$$
$$\left. + 2 \sum_{(j<k)=1}^{3} \frac{\partial^2 \varphi_2}{\partial \omega_j \partial \omega_k} \sum_{i=1}^{3} \frac{\partial \omega_j}{\partial x_i} \frac{\partial \omega_k}{\partial x_i} \right] f_2 + 2 \sum_{i=1}^{3} \frac{\partial f_2}{\partial x_i} \sum_{j=1}^{3} \frac{\partial \varphi_2}{\partial \omega_j} \frac{\partial \omega_j}{\partial x_i}$$
$$+ \varphi_2 \Delta f_2 + \Delta g_2 - a_2 [\varphi_2 f_2 + g_2] - [\varphi_1 f_1 + g_1]^n = 0 \tag{27}$$

where

$$\Delta := \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} .$$

To solve (26), (27) we need to find the functions ω, f_1, f_2, g_1, g_2. These functions are obtained by means of the Lie symmetry vector fields for system (8) listed above. Let us now consider system (8) with $a_1 \neq 0$ and $a_2 = 0$. In order to find ω and the functions f_1, f_2, g_1, g_2 we have to solve the following system:

$$\frac{\mathrm{d}x_1}{\xi_1(x,u)} = \frac{\mathrm{d}x_2}{\xi_2(x,u)} = \frac{\mathrm{d}x_3}{\xi_3(x,u)} = \frac{\mathrm{d}x_4}{\xi_4(x,u)} = \frac{\mathrm{d}u_0}{\eta_0(x,u)} = \frac{\mathrm{d}u_1}{\eta_1(x,u)} = \frac{\mathrm{d}\tau}{1}$$

where $u = (u_0, u_1)$, $\xi = (\xi_1, \xi_2, \xi_3, \xi_4)$ and $\eta = (\eta_0, \eta_1)$ are the coefficient functions of the Lie symmetry vector field with τ the Lie group parameter. We make use of the translations, rotations and scaling symmetry (11) to obtain

$$\omega_1(x) = \frac{\boldsymbol{\alpha}\cdot\boldsymbol{x}}{\sqrt{x_4}} \qquad \omega_2(x) = \frac{\boldsymbol{x}^2}{x_4} \qquad \omega_3(x) = -\ln x_4 + \tan^{-1}\left(\frac{\boldsymbol{\beta}\cdot\boldsymbol{x}}{\boldsymbol{\gamma}\cdot\boldsymbol{x}}\right)$$

$$f_1(x) = x_4 \qquad f_2(x) = x_4^{n+1} \qquad g_1(x) = 0 \qquad g_2(x) = 0$$

with

$$\boldsymbol{\alpha}^2 \equiv \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1 \qquad \boldsymbol{\beta}^2 \equiv \beta_1^2 + \beta_2^2 + \beta_3^2 = 1 \qquad \boldsymbol{\gamma}^2 \equiv \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1$$
$$\boldsymbol{\alpha}\cdot\boldsymbol{\beta} \equiv \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 = 0 \qquad \boldsymbol{\beta}\cdot\boldsymbol{\gamma} \equiv \beta_1\gamma_1 + \beta_2\gamma_2 + \beta_3\gamma_3 = 0$$
$$\boldsymbol{\gamma}\cdot\boldsymbol{\alpha} \equiv \gamma_1\alpha_1 + \gamma_2\alpha_2 + \gamma_3\alpha_3 = 0\,.$$

By inserting the above-obtained ω, f_1, f_2, g_1, g_2 into (26) and (27) we obtain the coupled system of partial differential

$$\frac{\partial^2\varphi_1}{\partial\omega_1^2} + 4\omega_2\frac{\partial^2\varphi_1}{\partial\omega_2^2} + (\omega_2 - \omega_1^2)^{-1}\frac{\partial^2\varphi_1}{\partial\omega_3^2} + 4\omega_1\frac{\partial^2\varphi_1}{\partial\omega_1\partial\omega_2}$$
$$-\frac{1}{2}\omega_1\frac{\partial\varphi_1}{\partial\omega_1} + (6-\omega_2)\frac{\partial\varphi_1}{\partial\omega_2} - \frac{\partial\varphi_1}{\partial\omega_3} + \varphi_1 - a_1 = 0 \tag{28}$$

$$\frac{\partial^2\varphi_2}{\partial\omega_1^2} + 4\omega_2\frac{\partial^2\varphi_2}{\partial\omega_2^2} + (\omega_2 - \omega_1^2)^{-1}\frac{\partial^2\varphi_2}{\partial\omega_3^2} + 4\omega_1\frac{\partial^2\varphi_2}{\partial\omega_1\partial\omega_2}$$
$$-\frac{1}{2}\omega_1\frac{\partial\varphi_2}{\partial\omega_1} + (6-\omega_2)\frac{\partial\varphi_2}{\partial\omega_2} - \frac{\partial\varphi_2}{\partial\omega_3} - \varphi_1^n(\omega) + (n+1)\varphi_2 = 0\,. \tag{29}$$

Let us assume that φ_1 depends only on ω_2. Then (28) takes the form

$$4\omega_2\frac{\mathrm{d}^2\varphi_1}{\mathrm{d}\omega_2^2} + (6-\omega_2)\frac{\mathrm{d}\varphi_1}{\mathrm{d}\omega_2} + \varphi_1 - a_1 = 0\,. \tag{30}$$

For arbitrary a_1 (30) has the special solution $\varphi_1(\omega_2) = a_1$. It follows, from the ansatz (24), that

$$u_0(x) = a_1 x_4\,. \tag{31}$$

For $a_1 = 6$ we obtain the special solution $\varphi_1(\omega_2) = \omega_2$ for (30) so that, from the ansatz (24), we have

$$u_0(x) = \boldsymbol{x}^2\,. \tag{32}$$

To find a special solution for (29) we assume that φ_1 and φ_2 depend only on ω_2 with $a_1 = 6$. From (29) it follows that

$$4\omega_2 \frac{d^2\varphi_2}{d\omega_2^2} + (6 - \omega_2)\frac{d\varphi_2}{d\omega_2} + (n+1)\varphi_2 - \omega_2^n = 0. \tag{33}$$

With $n = -2$ we obtain the solution

$$\varphi_2(\omega_2) = g(\omega_2)\omega_2^{-1/2} \exp\left(\tfrac{1}{4}\omega_2\right) \tag{34}$$

where g satisfies the ordinary differential equation

$$\frac{dg}{d\omega_2} = -\tfrac{1}{4}\omega_2^{-3/2} \exp\left(-\tfrac{1}{4}\omega_2\right) + c_1\omega_2^{-1/2} \exp\left(-\tfrac{1}{4}\omega_2\right)$$

with c_1 an arbitrary constant.

We can thus summarize as follows. For $n = -2$, $a_1 = 6$ and $a_2 = 0$ an approximate solution for (6) is given by

$$\begin{aligned} u(x) = x^2 + \varepsilon x_4^{-1}\Bigg[&-\omega_2^{-2} - \tfrac{1}{4}\omega_2^{-1/2} \sum_{\nu=1}^{N} (-1)4^{N+1} \frac{d^\nu}{d\omega_2^\nu}(\omega_2^{-3/2}) \\ &- \omega_2^{-1/2} 4^N \exp\left(\tfrac{1}{4}\omega_2\right) \int \frac{d^{N+1}}{d\omega_2^{N+1}}(\omega_2^{-3/2}) \exp\left(-\tfrac{1}{4}\omega_2\right) d\omega_2 \\ &+ 2c_1 \sum_{\nu=1}^{\infty} \frac{\omega_2^\nu}{2^\nu} \frac{1}{1 \times 3 \times \cdots \times (1+2\nu)} + c_2\Bigg]. \end{aligned} \tag{35}$$

Here c_1, c_2 are arbitrary constants, $\omega_2 = x^2/x_4$ and $N = 1, 2, \ldots$.

To find another solution for (29) we assume that φ_2 depends only on ω_1 with $\varphi_1(\omega_2) = a_1$. Here a_1 is arbitrary. From (29) it follows that

$$\frac{d^2\varphi_2}{d\omega_1^2} - \frac{1}{2}\omega_1 \frac{d\varphi_2}{d\omega_1} - \frac{1}{2}\varphi_2 - a_1^{-3/2} = 0 \tag{36}$$

where $n = -\frac{3}{2}$. We obtain the general solution of (36) as

$$\varphi_2(\omega_1) = \exp\left(\tfrac{1}{4}\omega_1^2\right)\left[c_1 \int \exp\left(-\tfrac{1}{4}\omega_1^2\right) d\omega_1 + c_2\right] - 2a_1^{-3/2} \tag{37}$$

where c_1 and c_2 are arbitrary constants.

We can thus summarize as follows. For $n = -\frac{3}{2}$, a_1 arbitrary and $a_2 = 0$ an approximate solution for (6) is given by

$$u(x) = x_4 + \varepsilon x_4^{-1/2}\left\{\exp\left(\tfrac{1}{4}\omega_1^2\right)\left[c_1 \int \exp\left(-\tfrac{1}{4}\omega_1^2\right) d\omega_1 + c_2\right] - 2a_1^{-3/2}\right\} \tag{38}$$

where $\omega_1(x) = (\alpha \cdot x)/\sqrt{x_4}$.

Let us now construct an exact solution for system (8) with n, a_1 and $a_2 \neq 0$ arbitrary. We make use of the translation symmetries and the Lie symmetry vector field (9) where f is given by (15) with $\alpha_1 = 1$ and $\alpha_2 = \alpha_3 = 0$, i.e.

$$Z_8 = x_1 \exp(a_2 x_4) \frac{\partial}{\partial u_1} .$$

Using the ansätze (24) and (25) as well as

$$\omega_1(x) = \boldsymbol{\alpha} \cdot \boldsymbol{x} \qquad \omega_2(x) = \boldsymbol{\beta} \cdot \boldsymbol{x} \qquad \omega_3(x) = x_4$$

$$f_1(x) = 1 \qquad f_2(x) = 1 \qquad g_1(x) = 0 \qquad g_2(x) = \frac{1}{a_2}\left(x_4 - \frac{1}{a_2}\right)\exp(a_2 x_4)$$

equations (26) and (27) reduce to

$$\frac{\partial^2 \varphi_1}{\partial \omega_1^2} + \frac{\partial^2 \varphi_1}{\partial \omega_2^2} + \frac{\partial \varphi_1}{\partial \omega_3} - a_1 - a_2 \varphi_1 = 0 \tag{39}$$

$$\frac{\partial^2 \varphi_2}{\partial \omega_1^2} + \frac{\partial^2 \varphi_2}{\partial \omega_2^2} + \frac{\partial \varphi_2}{\partial \omega_3} - a_2 \varphi_2 - \varphi_1^n + \frac{1}{a_2}\exp(a_2 \omega_3) = 0 . \tag{40}$$

Let us assume that φ_1 depends only on ω_3. With this assumption we obtain the general solution for (39) as

$$\varphi_1(\omega_3) = -\frac{a_1}{a_2} + c_1 \exp(a_2 \omega_3) .$$

Here c_1 is an arbitrary constant. From the ansatz (24) it follows that

$$u_0(x) = -\frac{a_1}{a_2} + c_1 \exp(a_2 x_4) .$$

Equation (40) reduces to the ordinary differential equation

$$\frac{\mathrm{d}\varphi_2}{\mathrm{d}\omega_3} - a_2 \varphi_2 + \frac{1}{a_2}\exp(a_2\omega_3) - \left(-\frac{a_1}{a_2} + c_1 \exp(a_2\omega_3)\right)^n = 0 \tag{41}$$

with the general solution

$$\varphi_2(\omega_3) = -\frac{1}{a_2}\omega_3 \exp(a_2\omega_3) - \left(-\frac{a_1}{a_2}\right)^n \left(\frac{1}{a_2}\right) + \left[c_1 {n \choose 1} \left(-\frac{a_1}{a_2}\right)^{n-1} \omega_3 \right.$$
$$\left. + \sum_{j=2}^{n} \frac{c_1^j}{(j-1)a_2} {n \choose j} \left(-\frac{a_1}{a_2}\right)^{n-j} \exp[(j-1)a_2\omega_3] + c_2\right] \exp(a_2\omega_3)$$

where c_2 is an arbitrary constant.

We can thus summarize as follows. For n, a_1 and $a_2 \neq 0$ arbitrary an approximate solution for (6) is given by

$$u(x) = -\frac{a_1}{a_2} + c_1 \exp(a_2 x_4) + \varepsilon \left\{ -\frac{1}{a_2^2}\exp(a_2 x_4) - \left(-\frac{a_1}{a_2}\right)^n \left(\frac{1}{a_2}\right) \right.$$
$$+ \left[c_1 {n \choose 1} \left(-\frac{a_1}{a_2}\right)^{n-1} x_4 + \sum_{j=2}^{n} \frac{c_1^j}{(j-1)a_2} {n \choose j} \left(-\frac{a_1}{a_2}\right)^{n-j} \right.$$
$$\left.\left. \times \exp[(j-1)a_2 x_4] + c_2\right] \exp(a_2 x_4) \right\} . \tag{42}$$

All approximate solutions obtained for the multidimensional Landau–Ginzburg equation (6) satisfy the equation up to order ε, since we have obtained these solutions from first-order approximate symmetries. This method of finding approximate solutions can obviously also be applied to other multidimensional nonlinear partial differential equations or systems. Approximate symmetries are of special importance when the starting equation or system does not admit interesting Lie symmetry vector fields.

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